

On the $SU(3)$ Parametrization of Qutrits¹

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Abstract

Parametrization of qutrits on the complex projective plane $\mathcal{CP}^2 = SU(3)/U(2)$ is given explicitly. A set of constraints that characterize mixed state density matrices is found.

Many recent ideas of quantum information theory are based on the notion of qubits. A qubit may be represented by a point on the Poincaré sphere S^2 that is homeomorphic to the complex projective line $\mathcal{H}^{(2)} = \mathcal{CP}^1 = SU(2)/U(1)$. A similar parametrization in the case of higher dimensional quantum systems is desirable both from theoretical [1] and technical points of view [2], [3]. A qutrit may be represented by a point on the complex projective plane $\mathcal{H}^{(3)} = \mathcal{CP}^2 = SU(3)/U(2)$. Such a representation is given explicitly in terms of Gell-Mann matrices [4]. We determine a set of constraints that characterize mixed states of qutrits below.

A qutrit $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle$ where $\alpha_0, \alpha_1, \alpha_2 \in \mathbf{C}$, $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$, is a state vector in the Hilbert space of states $\mathcal{H}^{(3)}$ of a 3-level system. It is spanned by an orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$ which in matrix notation reads

$$|0\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |2\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore

$$|\psi\rangle \rightarrow \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \in \mathbf{C}^3 \simeq \mathbf{R}^6.$$

Since $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$ and since $|\psi\rangle$ is determined up to a multiplicative phase factor, $\dim \mathcal{H}^{(3)} = 4$.

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Any 3×3 density matrix can be written as

$$\rho = \frac{1}{3}(I + \sqrt{3}\vec{n} \cdot \vec{\lambda})$$

where \vec{n} is a real 8-vector, and components of $\vec{\lambda}$ are the (Hermitian, traceless) Gell-Mann matrices

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\ \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

The product of two Gell-Mann matrices is given by

$$\lambda_j \lambda_k = \frac{2}{3} \delta_{jk} + \sum_l d_{jkl} \lambda_l + i \sum_l f_{jkl} \lambda_l$$

where $j, k = 1, 2, \dots, 8$. The f -symbols (structure constants of the Lie algebra $\mathfrak{su}(3)$) are totally anti-symmetric :

$$\begin{aligned} f_{123} &= 1, f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \\ f_{147} &= f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}, \end{aligned}$$

and the d -symbols are totally symmetric:

$$\begin{aligned} d_{118} &= d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, \quad d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \\ d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}. \end{aligned}$$

Given two real 8-vectors \vec{a} and \vec{b} , we define their Euclidean inner product

$$\vec{a} \cdot \vec{b} = \sum_k a_k b_k \quad ,$$

skew-symmetric vector \wedge -product

$$(\vec{a} \wedge \vec{b})_j = \sqrt{3} \sum_{k,l} f_{jkl} a_k b_l \quad ,$$

and symmetric vector \star -product

$$(\vec{a} \star \vec{b})_j = \sqrt{3} \sum_{k,l} d_{jkl} a_k b_l \quad .$$

The pure states that satisfy $\rho^2 = \rho$ are therefore characterized by

$$|\vec{n}|^2 = 1 \quad \text{and} \quad \vec{n} \star \vec{n} = \vec{n} \quad .$$

Suppose that $\rho = \frac{1}{3}(I + \sqrt{3}\vec{n} \cdot \vec{\lambda})$ is the density matrix of a mixed state. It is Hermitian, positive with trace equal to 1. Therefore all the eigenvalues x_1, x_2, x_3 are positive and add to one: $x_1 + x_2 + x_3 = 1$. The Cayley-Hamilton equation satisfied by ρ reads

$$\rho^3 - \rho^2 + (x_1 x_2 + x_2 x_3 + x_1 x_3)\rho - x_1 x_2 x_3 I = 0 \quad .$$

The following inequalities hold:

$$\frac{1}{3} \geq x_1 x_2 + x_2 x_3 + x_1 x_3 \geq 0 \quad , \quad \frac{1}{27} \geq x_1 x_2 x_3 \geq 0 \quad .$$

Starting from these, a straightforward computation shows that the necessary and sufficient conditions for $\rho = \frac{1}{3}(I + \sqrt{3}\vec{n} \cdot \vec{\lambda})$ to be a density matrix of a mixed state are given by

$$1 \geq |\vec{n}|^2 \geq 0 \quad \text{and} \quad 1 \geq 3|\vec{n}|^2 - 2\vec{n} \cdot (\vec{n} \star \vec{n}) \geq 0 \quad .$$

An arbitrary diagonal density matrix of a 3-level system will be

$$\rho = \frac{1}{3}(I + \sqrt{3}(n_3 \lambda_3 + n_8 \lambda_8)).$$

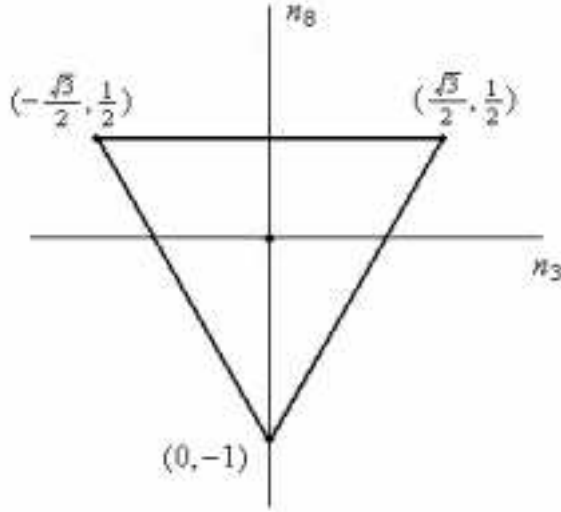
In this case, the mixed-state density matrix constraints reduce to

$$0 \leq n_3^2 + n_8^2 \leq 1 \quad \text{and} \quad 0 \leq 2n_8^3 - 6n_3^2 n_8 + 3n_3^2 + 3n_8^2 \leq 1 \quad .$$

The region in the $n_3 n_8$ -plane where both the constraints are satisfied is bound by an equilateral triangle with vertices at the points

$$(n_3, n_8)_R = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (n_3, n_8)_B = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(n_3, n_8)_G = (0, -1) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$



Vertices of the above triangle correspond to three mutually orthogonal pure-states. We labeled them Red, Blue, Green in analogy with colored quarks [2]. In fact two pure-state vectors $|\psi\rangle$ and $|\psi'\rangle$ are orthogonal if and only if $\langle \psi | \psi' \rangle = 0$, so that $\text{Tr}(\rho\rho') = 0$. This implies $\vec{n} \cdot \vec{n}' = -\frac{1}{2}$. Then $\arccos(\vec{n} \cdot \vec{n}') = \pm\frac{2\pi}{3}$. This is equal to the geodesic distance between two orthogonal pure-states as measured by the standard Fubini-Study metric on \mathcal{CP}^2 .

Points on the edges of the triangle correspond to mixing of two orthogonal pure-states of qutrits. In particular, at mid-points where bi-sectors intersect with the edges we have

$$(n_3, n_8)_C = (0, \frac{1}{2}) \leftrightarrow \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (n_3, n_8)_B = (\frac{\sqrt{3}}{4}, -\frac{1}{4}) \leftrightarrow \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(n_3, n_8)_A = (-\frac{\sqrt{3}}{2}, -\frac{1}{4}) \leftrightarrow \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Triply mixed-states correspond to points inside the triangle. In particular the origin

$$(n_3, n_8)_O = (0, 0) \leftrightarrow \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

corresponds to the maximally mixed state.

For 2-level systems, the orbit of any diagonal 2×2 density matrix of qubits

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + n_3 & 0 \\ 0 & 1 - n_3 \end{pmatrix} \leftrightarrow (0, 0, n_3)$$

under the action of the unitary group $\text{SU}(2)$,

$$\rho \rightarrow U \rho U^\dagger$$

where $U \in \text{SU}(2)$ (i.e. adjoint representation of $\text{SU}(2)$, which is $\text{SO}(3)$ applied on n_3) sweeps the whole Poincaré sphere S^2 . Adjoint representation Ad of a given $U \in \text{SU}(2)$ is explicitly

$$\text{Ad}(U)_{ij} = \frac{1}{2} \text{Tr}(\sigma_i U \sigma_j U^\dagger) \in \text{SO}(3),$$

so that $n_j \rightarrow \text{Ad}(U)_{jk} n_k$.

In a similar way, for 3-level systems the orbit of each point (n_3, n_8) of the above triangle under the unitary action of $\text{SU}(3)$ (i.e. adjoint representation of $\text{SU}(3)$) will provide a generalization of the Poincaré sphere to 3-level systems. Adjoint representation Ad of a given $U \in \text{SU}(3)$ is found as follows:

$$\text{Ad}(U)_{ij} = \frac{1}{2} \text{Tr}(\lambda_i U \lambda_j U^\dagger) \in \text{SO}(8)$$

so that

$$n_j \rightarrow \text{Ad}(U)_{jk} n_k.$$

In fact $\text{Ad}(\text{SU}(3))$ is an 8-parameter subgroup of the 28-parameter rotation group $\text{SO}(8)$.

We also consider the *entropy of mixing* of ρ defined as

$$E(\rho) = -x_1 \log_3(x_1) - x_2 \log_3(x_2) - x_3 \log_3(x_3) \quad .$$

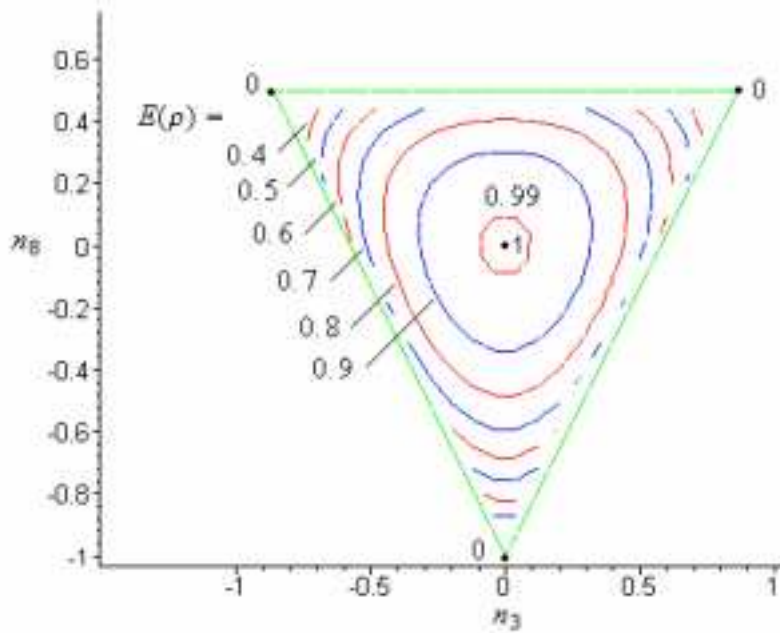
Since in diagonal form

$$\rho = \frac{1}{3} \begin{pmatrix} 1 + \sqrt{3}n_3 + n_8 & & \\ & 1 - \sqrt{3}n_3 + n_8 & \\ & & 1 - 2n_8 \end{pmatrix},$$

the entropy of mixing of ρ becomes

$$E(\rho) = - \left(\frac{1 + \sqrt{3}n_3 + n_8}{3} \right) \log_3 \left(\frac{1 + \sqrt{3}n_3 + n_8}{3} \right) - \left(\frac{1 - \sqrt{3}n_3 + n_8}{3} \right) \log_3 \left(\frac{1 - \sqrt{3}n_3 + n_8}{3} \right) - \left(\frac{1 - 2n_8}{3} \right) \log_3 \left(\frac{1 - 2n_8}{3} \right) \quad .$$

The equi-mixing curves in the $n_3 n_8$ -plane are shown on the following diagram:



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